JOURNAL OF MULTIVARIATE ANALYSIS 28, 9–19 (1989)

Independent Marginals of Infinitely Divisible and Operator Semi-stable Measures

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Communicated by T. Hida

Let μ be an infinitely divisible measure on a finite dimensional vector space. The problem of the existence and the uniqueness of independent martingals for μ is studied. A more detailed description is given for operator semi-stable measures. The results obtained generalize those proved for operator-stable measures by W. N. Hudson, J. D. Mason, and H. G. Tucker (1981, *Z. Wahrsch. Verw. Gebiete* **58** 285–297) and J. A. Veeh (1982, *Z. Wahrsch. Verw. Gebiete* **61** 303–308). © 1989 Academic Press, Inc.

1. INTRODUCTION AND PRELIMINARIES

Let V be a finite dimensional real vector space with an inner product $(\cdot, \cdot)$ and σ-algebra $\mathscr{B}(V)$ of its Borel subsets. A linear operator T on V is called a projection if $T^2 = T$. Two projections T, P are called orthogonal if $TP = PT = 0$.

Let $V = V_1 + V_2$ be a decomposition of V. The mapping $T_1 : V \to V_1$ defined as

$$T_1(v_1 + v_2) = v_1, \qquad v_1 \in V_1, \qquad v_2 \in V_2$$

is a projection called the projection on V_1 along V_2.

For a probability measure μ on V its characteristic function $\hat{\mu}$ is defined as

$$\hat{\mu}(v) = \int_V \exp\{i(u, v)\}\, \mu(du), \qquad v \in V.$$

Received May 15, 1986; revised July 20, 1987.

AMS 1980 subject classifications: Primary 60 B 11; Secondary 60 E 07.

Key words and phrases: marginals of a probability measure, infinitely divisible measures, operator semi-stable measures.

0047-259X/89 $3.00

Let $A: V \to W$ be a linear mapping into a finite dimensional real vector space W and let μ be a (probability) measure over $(V, \mathscr{B}(V))$. The measure $\mu_A = A\mu$ on $(W, \mathscr{B}(W))$ is defined by

$$A\mu(E) = \mu(A^{-1}(E)), \qquad E \in \mathscr{B}(W).$$

The following equalities are easily verified:

$$A(B\mu) = (AB)\mu, \qquad \widehat{A\mu}(v) = \hat{\mu}(A^*(v)), \qquad A(\mu * \nu) = A\mu * A\nu,$$

for linear operators A, B and probability measures μ, ν.

If λ is a Borel measure on V, then S_λ stands for the support of λ, i.e., the closed subset of V such that the complement of S_λ has λ-measure zero and $\lambda(U_v) > 0$ for each neighbourhood U_v of $v \in S_\lambda$. A measure is called full if its support is not contained in any proper hyperplane of V.

For a projection T in V and a probability measure μ on V, the measure $T\mu$ is called a marginal of μ. If $T\mu = \delta(0)$, the marginal is called trivial. If $\dim T(V) = 1$, the marginal is called univariate. A set $\{T_1\mu, ..., T_r\mu\}$ of marginals is complete if $\sum_{k=1}^r T_k = I$, where I is the identity mapping. The same name applies to the projections $T_1, ..., T_r$ alone.

Now, assume that $A_1, ..., A_r$ are arbitrary linear operators in V. $A_1, ..., A_r$ are said to be independent (with respect to μ) if they are independent as multivariate random variables from the probability space $(V, \mathscr{B}(V), \mu)$ to V. Since

$$\hat{\mu}_{A_1, ..., A_r}(v_1, ..., v_r) = \int_V \exp\left\{i \sum_{k=1}^r (A_k u, v_k)\right\} \mu(du),$$

this condition is equivalent to the equality

$$\hat{\mu}_{A_1, ..., A_r}(v_1, ..., v_r) = \prod_{k=1}^r \hat{\mu}_{A_k}(v_k), \qquad v_1, ..., v_k \in V. \tag{1}$$

The marginals $T_1\mu, ..., T_r\mu$ are called independent if the projections $T_1, ..., T_r$ are such.

We recall that an infinitely divisible measure μ on V has the unique representation $[m, D, M]$, where $m \in V$, D is a positive linear operator on V, and M is the Lévy spectral measure of μ, i.e., a Borel measure defined on $V_0 = V - \{0\}$, finite outside each neighbourhood of zero, and such that $\int_{\|v\| \leqslant 1} \|v\|^2 M(dv) < \infty$ (see, e.g., [6]).

For two infinitely divisible measures $\mu_1 = [m_1, D_1, M_1]$, $\mu_2 = [m_2, D_2, M_2]$, we have $\mu_1 * \mu_2 = [m_1 + m_2, D_1 + D_2, M_1 + M_2]$. If $\mu = [m, D, M]$ and A is a linear operator on V, then $A\mu$ is infinitely divisible and $A\mu = [m', ADA^*, AM | V_0]$ for some $m' \in V$ where, by a slight abuse of notation, we write $AM | V_0$ instead of $((A | V_0)M) | V_0$.

The problem of the existence of a complete set of independent univariate marginals for operator-stable measures was studied in [2, 7]. The aim of this paper is to carry out analogous investigations for operator semi-stable measures (being a generalization of operator-stable ones), as well as to make some contributions concerning the problem of marginals for an arbitrary infinitely divisible measure.

For the definition and a more detailed description of operator semi-stable measures, the reader is referred to [3, 4]. Here we recall only their very basic properties.

If μ is a full operator semi-stable measure on V, then μ is infinitely divisible and

$$\mu^a = A\mu * \delta(h) \tag{2}$$

for some $0<a<1$, $h \in V$, and a non-singular linear operator A in V. Measures satisfying (2) will be called (a, A)-quasi-decomposable. If μ is quasi-decomposable (by some pair (a, A)), then μ is operator semi-stable. Moreover, there are decompositions

$$\mu = \mu_N * \mu_P, \qquad V = V_N \oplus V_P \tag{3}$$

such that V_N and V_P are A-invariant subspaces of V, μ_N is a Gaussian measure concentrated on V_N, and μ_P is a purely Poissonian (a, A)-quasi-decomposable measure concentrated on V_P.

2. Independent Marginals: Infinitely Divisible Measures

We begin with the following generalization of Theorem 1 from [7].

THEOREM 1. *Let $\mu = [m, D, M]$ be an infinitely divisible measure on V and let $\{T_1, ..., T_r\}$ be a set of pairwise orthogonal projections. Put $T = \sum_{i=1}^r T_i$. Then $T_1, ..., T_r$ are independent if and only if*

(i) $TDT^* = \sum_{i=1}^r T_i D T_i^*$,

(ii) $T(S_M) \subset \bigcup_{i=1}^r T_i(V)$.

Proof. Let us observe first that the independence of $T_1, ..., T_r$ with respect to a given probability measure μ is equivalent to the representation

$$\mu_T = T_1 \mu * \cdots * T_r \mu. \tag{4}$$

Indeed, assume (4) and define a mapping $\mathscr{J} : V \to V^r$ by

$$\mathscr{J} v = (T_1 v, ..., T_r v), \qquad v \in V.$$

We have then

$$\mathscr{J}T = \mathscr{J} \qquad \text{and} \qquad \mathscr{J}T_i = (0, ..., T_i, ..., 0).$$

Consequently,

$$\mathscr{J}\mu_T = \mathscr{J}T\mu = \mathscr{J}\mu = \mu_{T_1, ..., T_r}$$

and putting $\mu_i = T_i\mu$ we obtain

$$\mathscr{J}\mu_i = \mathscr{J}T_i\mu = \delta(0) \times \cdots \times \mu_i \times \cdots \times \delta(0), \qquad i = 1, ..., r.$$

From the assumption we have

$$\begin{aligned}\hat{\mu}_{T_1, ..., T_r}(v_1, ..., v_r) &= \widehat{\mathscr{J}\mu_T}(v_1, ..., v_r) = \widehat{\mathscr{J}\mu_1 * \cdots * \mathscr{J}\mu_r}(v_1, ..., v_r) \\ &= \prod_{i=1}^{r} \widehat{\mathscr{J}\mu_i}(v_1, ..., v_r) = \prod_{i=1}^{r} \hat{\mu}_i(v_i)\end{aligned}$$

which proves that $T_1, ..., T_r$ are independent.

Conversely, if $T_1, ..., T_r$ are independent, then regarding each T_i as a multidimensional random variable on the probability space $(V, \mathscr{B}(V), \mu)$ we have

$$\mu_T = \mu_{T_1 + \cdots + T_r} = T_1\mu * \cdots * T_r\mu.$$

Now, the representation (4) is equivalent to the conditions

$$TDT^* = \sum_{i=1}^{r} T_i D T_i^*, \tag{5}$$

$$TM \mid V_0 = \sum_{i=1}^{r} (T_i M \mid V_0), \tag{6}$$

where we put $E_0 = E - \{0\}$ for a subset E of V. But (6) is equivalent to the inclusion

$$S_{TM \mid V_0} \subset \bigcup_{i=1}^{r} T_i(V)_0. \tag{7}$$

Indeed, if (7) holds, then

$$TM \mid V_0 = \sum_{i=1}^{r} (TM \mid T_i(V)_0) = \sum_{i=1}^{r} (T_i M \mid T_i(V)_0) = \sum_{i=1}^{r} (T_i M \mid V_0)$$

because $T_i(V)_0$ are pairwise disjoint. Clearly, (6) implies (7).

Now, $S_{TM|V_0}=\overline{T(S_M)_0}$ where the closure is in the space V_0. Indeed, we have

$$\begin{aligned}
&(TM|V_0)(V_0-\overline{T(S_M)_0})\\
&\quad\leqslant (TM)(V_0-T(S_M)_0)=M(T^{-1}(V_0)-T^{-1}(T(S_M)\cap V_0))\\
&\quad\leqslant M(T^{-1}(V_0)-(S_M\cap T^{-1}(V_0)))=0.
\end{aligned}$$

In order to show that $\overline{T(S_M)_0}$ is the support, take $v\in\overline{T(S_M)_0}$ and let U_v be its arbitrary neighbourhood in V_0. Since $v\neq 0$ (the closure is in V_0) we can find an open set U_1 such that $U_1\subset U_v$, $v\in U_1$, $0\notin U_1$. U_1 is a neighbourhood of v in V_0 as well, thus there exists a $v'\in T(S_M)_0\cap U_1$. Consequently, we can find a $z\in S_M$ such that $v'=Tz$. $T^{-1}(U_1)$ is a neighbourhood of z in V_0, so we have

$$(TM|V_0)(U_v)\geqslant (TM|V_0)(U_1)=M(T^{-1}(U_1))>0$$

which states the claim.

Consequently, (7) takes the form

$$\overline{T(S_M)_0}\subset\bigcup_{i=1}^{r}T_i(V)_0$$

which is equivalent to the inclusion

$$T(S_M)_0\subset\bigcup_{i=1}^{r}T_i(V_0) \tag{8}$$

because $T_i(V_0)$ are closed in V_0. Clearly, (8) is equivalent to (ii). ▌

Let us consider now the problem of the uniqueness of independent orthogonal univariate marginals for an infinitely divisible measure. It is known that if μ is Gaussian, full, and dim $V\geqslant 2$, then μ has infinitely many independent pairwise orthogonal univariate marginals. Thus the uniqueness problem makes sense only for a measure with "sufficiently large" Poissonian part.

THEOREM 2. *Let $\mu=[m,D,M]$ be an infinitely divisible measure on V such that the measure $[m,0,M]$ is full. If μ has a complete set of independent pairwise orthogonal univariate marginals, then these marginals are unique.*

Proof. Let us assume that $\{T_1\mu,\dots,T_r\mu\}$ and $\{P_1\mu,\dots,P_r\mu\}$ are two complete sets of independent pairwise orthogonal univariate marginals. From Theorem 1 with $T=\sum_{i=1}^{r}T_i=\sum_{m=1}^{r}P_m=I$, we obtain

$$S_M\subset\bigcup_{i=1}^{r}T_i(V)\cap\bigcup_{m=1}^{r}P_m(V).$$

Take an arbitrary T_i. We have

$$S_M \cap T_i(V) = \bigcup_{m=1}^{r} (S_M \cap T_i(V) \cap P_m(V))$$

and $S_M \cap T_i(V) \neq \emptyset$ since M is full. It follows that $S_M \cap T_i(V) \cap P_{m(i)}(V) \neq \emptyset$ for some $m(i)$ and, consequently, $T_i(V) = P_{m(i)}(V)$, since $T_i(V)$, $P_{m(i)}(V)$ are one-dimensional and $0 \notin S_M$. This implies that

$$T_j P_{m(i)} = P_{m(i)} T_j = 0, \qquad i \neq j,$$

and thus

$$T_i = \sum_{j=1}^{r} T_i P_{m(j)} = T_i P_{m(i)},$$

$$P_{m(i)} = \sum_{j=1}^{r} T_j P_{m(i)} = T_i P_{m(i)}$$

which gives $T_i = P_{m(i)}$, proving the theorem. ▌

For an arbitrary probability measure on V a uniqueness result is also possible, namely, we have

PROPOSITION 3. *Let μ be a probability measure on V. Assume that there are decompositions*

$$\begin{aligned} \mu &= \mu_1 * \mu_2 \qquad & V &= V_1 \oplus V_2 \\ \mu &= \nu_1 * \nu_2 \qquad & V &= V_1 \oplus W_2 \end{aligned} \tag{9}$$

such that μ_1, ν_1 are concentrated on V_1, μ_2 is concentrated on V_2, ν_2 is concentrated on W_2, and either μ_2 or ν_2 (or both) is full on V_2 or W_2, respectively (the last assumption is satisfied if, for instance, μ is full). Then $V_2 = W_2$, $\mu_1 = \nu_1$, and $\mu_2 = \nu_2$.

Proof. Assume that μ_2 is full on V_2, and let T_1 be the projection on V_1 along V_2, T_2 be the projection on V_2 along V_1, P_1 be the projection on V_1 along W_2, and P_2 be the projection on W_2 along V_1. After symmetrization, we obtain from (9) the equalities

$$\mu^s = \mu_1^s * \mu_2^s = \nu_1^s * \nu_2^s.$$

In particular, we have $\mu_1^s = T_1 \mu^s$, $\mu_2^s = T_2 \mu^s$, $\nu_1^s = P_1 \mu^s$, $\nu_2^s = P_2 \mu^s$. Thus, we get

$$T_1 \mu^s = T_1 P_1 \mu^s * T_1 P_2 \mu^s = P_1 \mu^s * T_1 P_2 \mu^s,$$

$$P_1 \mu^s = P_1 T_1 \mu^s * P_1 T_2 \mu^s = T_1 \mu^s * P_1 T_2 \mu^s$$

and, consequently,

$$T_1\mu^s = T_1\mu^s * P_1T_2\mu^s * T_1P_2\mu^s.$$

Hence

$$P_1T_2\mu^s * T_1P_2\mu^s = \delta(0),$$

which is easily seen to be equivalent to the conditions

$$P_1T_2\mu^s = \delta(h),$$
$$T_1P_2\mu^s = \delta(-h)$$

for some $h \in V$. But $P_1T_2\mu^s = (P_1T_2\mu)^s$; thus the measure $P_1T_2\mu^s$ is symmetric, which implies $h = 0$.

We have thus obtained that $P_1\mu_2^s = P_1T_2\mu^s = \delta(0)$. Now, $W_2 = P_1^{-1}(\{0\})$ implies $\mu_2^s(W_2) = 1$. μ_2^s is concentrated on V_2 and therefore $\mu_2^s(W_2 \cap V_2) = 1$. But $W_2 \cap V_2$ is a subspace of V_2 and since μ_2^s is full on V_2 we get $W_2 \cap V_2 = V_2$, which means that $V_2 \subset W_2$. This inclusion, together with decompositions (9), imply the equality $V_2 = W_2$. Thus $T_1 = P_1$, $T_2 = P_2$, and, since $\mu_1 = T_1\mu$, $\nu_1 = P_1\mu$, $\mu_2 = T_2\mu$, $\nu_2 = P_2\mu$, the proof has been finished. ∎

As an immediate consequence of the above proposition and independence condition (1) we obtain

Corollary 4. *Let μ be a full probability measure on V and let $\{T_1, T_2\}$ and $\{P_1, P_2\}$ be two complete sets of independent pairwise orthogonal projections. If $T_i(V) = P_j(V)$ for some $i, j \in \{1, 2\}$, then $\{T_1, T_2\} = \{P_1, P_2\}$.*

We are now able to prove the following generalization of the main result of [2], which applies to operator-stable as well as to operator semi-stable measures.

Theorem 5. *Let $\mu = [m, D, M]$ be a full infinitely divisible measure on V. Assume that there are decompositions*

$$\mu = \mu_N * \mu_P \qquad V = V_N \oplus V_P \tag{10}$$

such that μ_N is a Gaussian measure concentrated on V_N and μ_P is a purely Poissonian measure concentrated on V_P.

Let F_N be the projection on V_N along V_P, F_P be the projection on V_P along V_N, and let $\{T_1, \ldots, T_r\}$ be a complete set of pairwise orthogonal one-dimen-

sional projections. Put $\mathscr{T}_N = \{i\colon \dim T_i(V_N) = 1\}$, $\mathscr{T}_P = \{i\colon \dim T_i(V_P) = 1\}$. *Then the marginals* $T_1\mu, \ldots, T_r\mu$ *are independent if and only if*

(i) $F_N T_i = \begin{cases} T_i & \text{for } i \in \mathscr{T}_N \\ 0 & \text{for } i \notin \mathscr{T}_N \end{cases}$ $\quad F_P T_i = \begin{cases} T_i & \text{for } i \in \mathscr{T}_P \\ 0 & \text{for } i \notin \mathscr{T}_P \end{cases}$;

(ii) $D = \sum_{i \in \mathscr{T}_N} T_i D T_i^*$;

(iii) $S_M \subset \bigcup_{i \in \mathscr{T}_P} T_i(V)$.

Proof. Let us assume that $T_1, \ldots, T_r$ are independent and put, for a moment, $\mathscr{I} = \{i\colon S_M \cap T_i(V) \neq \varnothing\}$. We have $S_M \subset \bigcup_{i \in \mathscr{I}} T_i(V)$ and

$$\operatorname{Lin} S_M = \bigoplus_{i \in \mathscr{I}} T_i(V) = V_P,$$

since μ_P is full on V_P. Let $F_1 = \sum_{i \in \mathscr{I}} T_i$, $F_2 = I - F_1$. From the independence of $T_1, \ldots, T_r$ it follows that F_1, F_2 are independent as well, thus $\mu = F_1\mu * F_2\mu$ by virtue of (1). On the other hand, $\mu = F_N\mu * F_P\mu$ and since $F_1(V) = V_P$ we infer, taking into account Corollary 4, that $F_1 = F_P$, $F_2 = F_N$.

Now, it is easily seen that $\mathscr{I} = \mathscr{T}_P$ and therefore $F_P = \sum_{i \in \mathscr{T}_P} T_i$, $F_N = \sum_{i \in \mathscr{T}_N} T_i$ so (i) follows.

Since μ_N is concentrated on V_N and μ_P is purely Poissonian, we have $D(V) = V_N$; thus $T_i D T_i^* = 0$ for $i \in \mathscr{T}_P$, which gives (ii). That (iii) holds has been noticed at the beginning of the proof.

Assume now that (i), (ii), and (iii) hold. We have

$$F_P = F_P \sum_{i=1}^{r} T_i = \sum_{i \in \mathscr{T}_P} T_i.$$

For $i \in \mathscr{T}_P$, $T_i D T_i^* = 0$ because $F_P D = 0$ and from (ii) we get

$$D = \sum_{i=1}^{r} T_i D T_i^*.$$

Clearly, $S_M \subset \bigcup_{i=1}^{r} T_i(V)$ which, according to Theorem 1, ends the proof. ∎

3. Independent Marginals: Operator Semi-Stable Measures

Our final aim is to give a more detailed description of the case when the measure considered is purely Poissonian operator semi-stable. In view of decompositions (3), this setting seems to be quite natural and, in many cases, it is enough to describe the Poissonian part alone.

THEOREM 6. *Let $\mu=[m,0,M]$ be an (a,A)-quasi-decomposable ($a>0$, $a\neq 1$) probability measure on V, and let $\{T_1,...,T_r\}$ be a complete set of independent pairwise orthogonal one-dimensional projections. Then*

(i) *$T_i\mu$ is an univariate semi-stable measure for $1\leqslant i\leqslant r$;*

(ii) *there exists a positive integer m such that for some $\alpha_i\in\mathbb{R}$,*

$$A^m=\sum_{i=1}^{r}\alpha_i T_i.$$

Proof. First, let us observe that, by iterating (2), we obtain that μ is (a^n, A^n)-quasi-decomposable for all n; in particular, the Lévy spectral measure M satisfies

$$a^n M = A^n M, \qquad n=0,\pm 1,... . \tag{11}$$

Let us prove (ii). We have $S_M\subset\bigcup_{i=1}^r T_i(V)$ and for any fixed $1\leqslant j\leqslant r$ take $v_j\in S_M\cap T_j(V_0)$. S_M is A^n-invariant by (11); thus $\{A^n v_j: n=0,\pm 1,...\}\subset S_M$. It follows that there exist $1\leqslant i\leqslant r$, s,k-positive integers, $s>k$, such that $A^s v_j$ and $A^k v_j$ belong to $T_i(V_0)$. To simplify the notation, put for a moment $A^k v_j=w$, $A^{s-k}=B$. Then $T_i(V)=\{\lambda w:\lambda\in\mathbb{R}\}$ because $\dim T_i(V)=1$, and $B(\lambda w)=\lambda Bw=\lambda A^s v_j\in T_i(V)$. Thus $T_i(V)$ is B-invariant, B is linear, and, consequently,

$$B\,|\,T_i(V)=\gamma_j I\,|\,T_i(V).$$

From this equality we get

$$A^s v_j = Bw = \gamma_j w = A^k(\gamma_j v_j)$$

so that

$$A^{s-k}v_j=\gamma_j v_j.$$

Denoting $m_j=s-k$, we finally obtain that, for each $1\leqslant j\leqslant r$, there exist a positive integer m_j and real number γ_j such that

$$A^{m_j}v_j=\gamma_j v_j.$$

If we now put

$$m=m_1\cdots m_r, \qquad \alpha_j=\gamma_j^{m/m_j}, \qquad j=1,...,r,$$

we shall get

$$A^m v_j=\alpha_j v_j,$$

which, on account of the equality $T_j v_j = v_j$ and the fact that $\{v_1, ..., v_r\}$ is a basis in V, proves (ii).

Now, we prove (i). Let us observe that

$$T_i A^m = A^m T_i = \alpha_i T_i, \qquad i = 1, ..., r.$$

The measure μ is (a^m, A^m)-quasi-decomposable and we have

$$\mu^{a^m} = A^m \mu * \delta(h').$$

Consequently,

$$(T_i \mu)^{a^m} = T_i A^m \mu * \delta(h_i) = (\alpha_i I)(T_i \mu) * \delta(h_i).$$

The last equality says that the univariate measure $T_i \mu$ is quasi-decomposable by a^m and the multiple of the identity; thus it is semi-stable, which ends the proof. ▮

Now, we turn to some uniqueness problems. Let us recall that if μ is quasi-decomposable by a pair (a, A), then it is quasi-decomposable by the pair (a^n, A^n) for every integer n. For an arbitrary full operator semi-stable measure μ, the sequence $\{a^n : n = 0, \pm 1, ...\}$ is, in some sense, unique (cf. [5, Theorem 3.2]). If, in addition, we assume that μ possesses complete orthogonal independent marginals, then we obtain the following "uniqueness" result.

THEOREM 7. *Let $\mu = [m, 0, M]$ be an (a, A)-quasi-decomposable $(a > 0, a \neq 1)$ probability measure on V having a complete set of independent univariate marginals. If μ is quasi-decomposable by a pair (a, B), then there exists a positive interger n such that $A^n = B^n$.*

Proof. According to Theorem 6 we have

$$A^m = \sum_{i=1}^{r} \alpha_i T_i, \qquad B^k = \sum_{i=1}^{r} \beta_i T_i$$

for some positive integers m, k. μ is (a^{mk}, A^{mk})- and (a^{mk}, B^{mk})-quasi-decomposable. Put

$$C = A^{mk} B^{-mk} = \sum_{i=1}^{r} \alpha_i^k \beta_i^{-m} T_i.$$

Then

$$\mu = C\mu * \delta(h'), \qquad h' \in V$$

and by [1, Theorem 5],

$$C = WRW^{-1},$$

where R is orthogonal and W is positive. Thus we have

$$|\alpha_i^k \beta_i^{-m}| = 1, \qquad i = 1, ..., r,$$

because $\alpha_i^k \beta_i^{-m}$ are eigenvalues of C. If we now take $n = 2mk$, we shall get

$$A^n = \sum_{i=1}^{r} \alpha_i^{2k} T_i = \sum_{i=1}^{r} \beta_i^{2m} T_i = B^n,$$

which concludes the proof. ▮

As a corollary we obtain the following fact proved in [2]:

COROLLARY 8. *If $\mu = [m, 0, M]$ is a full operator-stable measure having a complete set of independent univariate marginals, then its exponent is unique.*

Proof. Assume that A and B are exponents of μ, i.e.,

$$\mu^t = t^A \mu * \delta(h_t') = t^B \mu * \delta(h_t''), \qquad t > 0.$$

Fix an arbitrary $t_0 > 0$, $t_0 \neq 1$. μ is (t_0, t_0^A)- and (t_0, t_0^B)-quasi-decomposable, thus

$$(t_0^A)^n = (t_0^B)^n$$

for some n. Consequently, $nA = nB$ and $A = B$. ▮

ACKNOWLEDGMENTS

The author is indebted to Dr. J. Domsta for valuable comments on a preliminary draft of this paper.

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